Research Proposal Summary

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Take three numbers, a, b and c. If we add them together, it doesn't matter where we put the brackets, because $(a+b)+c = a+(b+c)$ regardless of what a, b and c actually are. Similarly, it doesn't matter where we put the brackets when we multiply them, because $(ab)c = a(bc)$. We say that addition and multiplication are both *associative* operations. Associative operations are very common in mathematics, so we have a name for a set with an associative binary operation on it – we call such a structure a *semigroup*. So $(\mathbb{R}, +)$ and $(\mathbb{R}, \times)$ are both semigroups (where $\mathbb{R}$ is the set of real numbers). The theory of semigroups is an example of an *algebraic theory*: a collection of formal operations and equations they are required to satisfy. Mathematicians study many such theories: the study of theories as objects in their own right is called *universal algebra*.

Two elements of a set are either equal or not: there is no middle ground. If we replace sets with categories, however, things become more complicated. Categories were invented to model the ways in which things are connected: thus, it's possible for two objects in a category to be unequal, but connected by a very strong sort of connection called an *isomorphism*. This means that they're indistinguishable for all practical purposes, but nonetheless actually different. When we try to study categories with algebraic structure, we find that most of the examples that occur in nature do not satisfy equations like associativity strictly, but only "up to isomorphism". In the case of associativity, we might have a category $\mathcal{C}$ and a binary operation $\otimes$ on $\mathcal{C}$, and $(a \otimes b) \otimes c$ would be isomorphic to $a \otimes (b \otimes c)$ for all a, b and c in $\mathcal{C}$. But $(a \otimes b) \otimes c$ would not actually be equal to $a \otimes (b \otimes c)$ in general. For example, in the category of sets, we find that the Cartesian products $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic, but they are not equal: one is $\{((a, b), c) : a \in A, b \in B, c \in C\}$, and the other is $\{(a, (b, c)) : a \in A, b \in B, c \in C\}$.

It turns out, for all theories that have so far been studied, that categories with algebraic structure are better behaved than this suggests: the isomorphisms themselves turn out to satisfy equations which make calculations manageable. But it is not obvious how to derive the equations satisfied by the isomorphisms from the equations in the original theory. This is the problem I have been attempting to solve. I have developed a way of constructing the categorified, "up to isomorphism" version of a theory, provided that the equations of that theory obey some syntactic constraints. This generalises various categorifications that had already been performed: in my paper [1], I show that a well-known theorem for monoidal categories is true in this more general setting. In my paper [2], I show that it doesn't matter how one describes the theory (which equations and operations one takes as primitive): the categorified version will be the same.

As part of Dalhousie's world-class category theory group, I would attempt to remove the syntactic constraints on the theories I can categorify, and investigate the question of finite presentability for categorified theories. I would also apply category theory and universal algebra to problems in computer science, in collaboration with Peter Selinger: in particular, I am interested in understanding the semantics of array-based programming languages like APL and its descendents.

References

[1] Miles Gould. Coherence for categorified operadic theories. Available as arXiv:0607423, 2006.

[2] Miles Gould. The categorification of a symmetric operad is independent of signature. Submitted to Applied Categorical Structures, available as arXiv:0711.4904, 2007.