Research Proposal

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1 Introduction

I have been attempting to construct a general theory of weak T-categories for any finitary algebraic theory T. The main motivating examples are the theories of weak monoidal categories and weak symmetric monoidal categories. These are categories with tensor products and units, where associativity and unit axioms (and, in the symmetric case, commutativity axioms) hold up to isomorphism: i.e. $(A \otimes B) \otimes C$ is isomorphic to $A \otimes (B \otimes C)$ and $A \otimes I \cong I \otimes A \cong A$ (and, in the symmetric case, $A \otimes B \cong B \otimes A$).

This question has been considered in the special cases of monoids, commutative monoids, groups, Lie algebras, rigs (= semirings), and crossed monoids, but the answer is not known in general. The problem is simpler to explain in the case of one-sorted theories; in this special case, the problem is as follows:

A *one-sorted algebraic theory* is given by some set of finitary operators and equations between them. For instance, the theory of groups is given by m (multiplication), u (unit), i (inversion) and all trees of these operators, and the equations

$$m \circ (1 \times m) = m \circ (m \times 1)$$
$$m \circ (1 \times u) = m \circ (u \times 1) = 1$$
$$m \circ (1 \times i) \circ d = m \circ (i \times 1) \circ d = u \circ !$$

and all consequences of these (where d is the diagonal operator, $d(x) = (x, x)$, and ! is the unique map $A \to A^0 = 1$). The **algebras** for such a theory are sets A equipped with a function $f_p : A^n \to A$ for each n-ary operator p, satisfying the equations.

A weak T-category, for some one-sorted theory T presented by operators $p_1, p_2, \ldots$ and equations $e_1, e_2, \ldots$, ought to be

- a category $\mathcal{C}$
- for each n-ary operator p, a functor $F_p : \mathcal{C}^n \to \mathcal{C}$
- for each equation e_i, a natural isomorphism α_i from the functor described on the left-hand side of e_i to that on the right-hand side of e_i.

The difficulty is that we do not want too many of these "coherence isomorphisms": we want them to be "coherent". Roughly, we want there to be exactly one isomorphism witnessing every distinct proof that two terms are the same. So we must find equations satisfied by the coherence isomorphisms themselves, such as the associativity pentagon found in the definition of weak monoidal category. These "coherence axioms" are typically hard to find and unmemorable. Having found them, the first order of business is usually to prove an "all diagrams commute" coherence theorem, but it's not clear what such a theorem should even say for general T.

2 Approach

The approach I have been considering makes use of enriched operads. Plain operads (i.e., those not equipped with a symmetric group action) can only describe so-called strongly regular theories: those where the equations involve the same variables on each side, each appearing only once and in the same order on both sides (so the theory of monoids is strongly regular, but that of commutative monoids is not because of the equation $a.b = b.a$, and that of groups is not because of the equation $a.a^{-1} = 1$). It is conjectured (though not, as far as I know, proven) that the theories described by symmetric operads satisfy a similar syntactic condition: that each variable in an equation appears exactly once on each side (though the order may vary from side to side). Let us assume for now that this is the case. Such theories are called *linear*, by analogy with linear logic.

Let T be a linear theory given by the symmetric operad P. Let Φ be a generating sequence for P, which is to say that there exists a regular epi

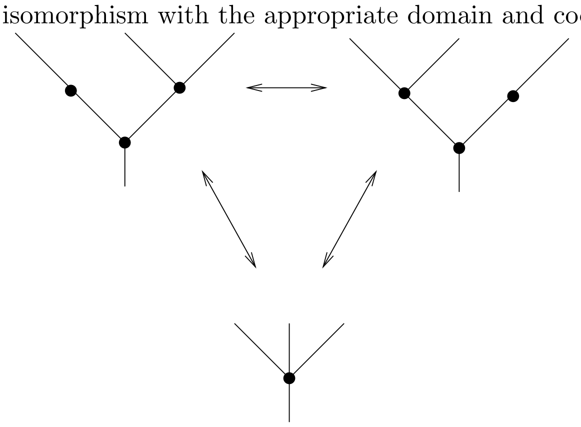
$$F\Phi \xrightarrow{\phi} P$$

where $U : \Sigma - \mathbf{Operad} \to \mathbf{Set}^{\mathbb{N}}$ is the forgetful functor, and F is its left adjoint. There is a factorization system on $\mathbf{Cat} - \Sigma - \mathbf{Operad}$, which allows us to factor ϕ uniquely as a bijective-on-objects map followed by a levelwise full-and-faithful one:

$$\begin{array}{ccc} F\Phi & \xrightarrow{\phi} & P \\ & \searrow \quad \nearrow & \\ & Q & \end{array}$$

A *weak T-category* with respect to ϕ is an algebra for the **Cat**-operad Q. In particular, an *unbiased weak T-category* is a weak T-category with respect to the counit of the adjunction $F \dashv U$. In this case, Q is given as follows:

The objects of the categories $Q(i)$ are elements of $FUP(i)$. In other words they are i-leafed permuted trees, with each n-leafed branch point labelled with an element of $P(n)$ (an *n-leafed permuted tree* is an n-leafed tree together with an element of S_n, which is applied to the leaves of the tree). We add an isomorphism between every pair of permuted trees which can be shown to be equal as a consequence of the equations, and define the composite of two isomorphisms to be the unique isomorphism with the appropriate domain and codomain.



Similarly, we can define notions of weak T-functor and T-transformation.

When T is the theory of monoids, an unbiased weak T category is exactly a weak unbiased monoidal category, which is essentially the same as a classical weak monoidal category. When T is the theory of commutative monoids, a weak T-category with respect to the standard presentation is exactly a symmetric monoidal category. The definitions presented can be applied when P is a symmetric multicategory, and here we find more examples: for instance, categorifying the multicategory whose algebras are maps of algebras for some operad P' yields the theory of weak P' functors.

3 Results

Armed with these definitions, we can prove the following theorems:

Theorem 1. *Let T be a strongly regular theory. Then every unbiased weak T-category is equivalent via weak T-functors and T-transformations to a strict T-category.*

This generalizes Joyal & Street's result for T = Monoid, or more accurately Leinster's result for unbiased monoidal categories, and the proof is essentially the same. In fact, the "strictification" process needed for the proof has an interesting universal property: the functor $\mathbf{st} : \mathbf{Wk}\text{-}T\text{-}\mathbf{Cat} \to \mathbf{Str}\text{-}T\text{-}\mathbf{Cat}$ is left adjoint to the forgetful functor $\mathbf{Str}\text{-}T\text{-}\mathbf{Cat} \to \mathbf{Wk}\text{-}T\text{-}\mathbf{Cat}$.

This result was presented at the 83rd Peripatetic Seminar on Sheaves and Logic; it is contained in my paper *Coherence for Categorified Operadic Theories.*

Theorem 2. *Let P be a symmetric operad, and let $\phi : F\Phi \to P$ be a regular epi. Then $Wk_\phi(P) \simeq Wk_\epsilon(P)$, where ϵ is the counit of $F \dashv U$.*

This implies that the category of weak P-categories w.r.t. ϕ is equivalent to the category of weak P-categories w.r.t. any other regular epi. This result was presented at CT 2007; a paper, *The Categorification of a Symmetric Operad is Independent of Signature*, has been submitted to *Applied Categorical Structures.*

The remaining results concern the relationship between my work and Blackwell, Kelly and Power's notion of a "pseudo-algebra for a 2-monad". Briefly, a pseudo-algebra for a 2-monad T satisfies up to coherent isomorphism the usual axioms for an algebra for a monad. Pseudo-algebras are often suggested as an approach to categorification, because of the following result:

Theorem 3 (Not new)**.** *Let T be a strongly-regular theory. Lift this to a 2-monad T' along the "discrete category" functor. Then the category of pseudo-algebras for T' in **Cat** is isomorphic to the category of weak T-categories.*

This is well-known in the case $T = \text{Monoid}$, and somewhat known in general. However, attempting to generalise this to the linear case or beyond leads to difficulties:

Theorem 4. *The pseudo-algebras in **Cat** for the "free commutative monoid" monad are the strictly symmetric weak monoidal categories, i.e. those for which $A \otimes B = B \otimes A$.*

Theorem 5. *Symmetric monoidal categories are pseudo-algebras for the "free symmetric strict monoidal category" 2-monad.*

While this does give us a way of understanding symmetric monoidal categories, the need to alter the 2-monad seems forced and unnatural. Compare the way in which symmetric monoidal categories arise naturally from the definition in Section 2.

4 Further Work

Dalhousie has a world-class category theory group, and I hope to continue my research in this environment, working with such noted category theorists as Bob Paré, Richard Wood, Peter Selinger and Dorette Pronk. In particular, I want to generalise my construction to theories that are not linear, and see how far Theorem 1 can be generalised. I also intend to investigate the question of finite presentability for categorified theories: in each case so far considered, the coherence axioms can be expressed as a finite set of diagrams. Is this the case for every finitely-presented theory? There are known counterexamples (due to Jonathan Cohen) when the coherence morphisms are not required to be invertible, but the answer is not known if they are.

I would also like to work on problems not directly related to my PhD topic: in particular, I would like to work with Peter Selinger on the application of category theory to computer science. For instance, by applying the theory of enriched categories to the "shape polymorphism" exhibited by programming languages such as APL, I hope to understand the semantics of these under-appreciated languages. I also have a more speculative project in the field of computational Assyriology: by statistical analysis of the corpus of known texts, it may be possible to determine the metric structure of poetry in the ancient Middle Eastern language Sumerian, and thus gain an insight into its pronunciation and prosody.